Lev Genrikhovich Schnirel'mann
(Лев Генрихович Шнирельман)

by
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January 1995 marks the 90^{th} anniversary of the birthday of L. G. Schnirel'mann, one of the most remarkable mathematicians of 1930's.

Until the present day we do not fully comprehend one of the most amazing miracles of the twentieth century: The Moscow school of mathematics. In 1914 there was only one internationally recognized mathematician working in Moscow. This was Dmitrii Fedorovich Egorov, who was working in a narrow specialty of differential geometry. At the same time France had giants such as Poincare, Picard, Hadamard, Lebesgue, Borel; Germany was the home to Klein, Hilbert, Weyl, all of them mathematicians of the broadest range. They belonged to the most distinguished schools of mathematics in centuries!

And then, so quickly, twenty years passed, seven of which were filled with wars that devastated our country, a blink of an eye in history. But, when in the mid-thirties, a well known American mathematician S. Lefshetz was asked who, in his opinion, were the most remarkable young mathematicians, he named four: Gel'fond, the solver of one of the Hilbert's problems, Kolmogorov, who made outstanding contributions to the theory of trigonometric series, probability, topology, and many other areas of mathematics, Pontriagin, discoverer of the new areas in topology and analysis, and Schnirel'mann, who obtained striking results in number theory, topology, and variational calculus.

In addition, one should mention the brilliant figures such as Urysohn, who died at the age of 26, Alexandrov, Bari, P. Novikov, Lavrentiev, Lyusternik, Petrovskiî, Khinchin, all of whose golden period of creativity happened during these years. Furthermore, those were the years when Gel'fand was taking his first steps . . .

However, in this bouquet of remarkable talent, the accomplishments of Schnirel'mann shine the brightest.

L. G. Schnirel'mann was born on January 2, 1905, in the town of Gomel. His father was a teacher of Russian language.

Lev Genrikhovich, early on, has shown to have an outstanding talent. He drew, wrote poetry, and at the age of 12 independently and on his own completed a course in elementary mathematics. For a period of several months the youngster attended courses in physics and mathematics given for high school graduates. There, he attracted attention of the instructors, to the point that they arranged for the youngster to be sent to Moscow for further studies.

Lev Genrikhovich Schnirel'mann

At the age of 15 he already flexed his muscles in an independent work. According to the legend (these always accompany biographies of outstanding people), he arrived in

Moscow at the age of 16 and enrolled in the Moscow (State, tr.) University. He brought with him notes, written in a school exercise book (made of cheap paper, better quality was not available during those difficult times), which contained a theorem about partitioning of a sphere. We will discuss this theorem later on, suffice it to say now is that it played a crucial role in the solution of the Poincare problem about closed geodesics – the solution that was found later by Schnirelman, jointly with Lazarev Aronovitch Liusternik. Solving this problem made Schnirel'mann known throughout the world.

Upon completing his studies at the university in two and a half years, Schnirel'mann was accepted as an aspirant at the Institute of Mathematics and Mechanics at the MGU. Like almost all of the mathematicians mentioned by us, he was a a student of Nikolai Nikolaevitch Lusin, the only exceptions being Pontriagin, who was a student of Alexandrov, and Petrovski, who was a student of Egorov.

Lazar Aronovich (Liusternik, tr.) used to reminisce that Luzin, (who apparently was inclined to view the world in a somewhat mystical way), had a dream in which a young man ("with the same biography as Lev Genrikhovich," the way L. A. put it), came to him and solved the continuum hypothesis. Then, when young Schnirel'mann actually appeared on the scene, Liusternik viewed him as a messenger from God. But Schnirel'mann did not solve the continuum hypothesis, the solution of that problem had to wait more than sixty years, when it was found by Paul Cohen. (It was actually only 35, tr.)

Schnirel'mann published three of his most remarkable papers during two years: in 1929 and 1930. Here are the statements of these theorems:

Theorem 1. *Inside an arbitrary closed curve in the plane, one can inscribe a square*.

Consider a thin piece of thread, with both ends tied together. This results in a "closed curve." We toss this curve on a table, and we are guaranteed that one can find four points on the curve, forming the vertices of a square.

Theorem 2. *On an arbitrary smooth surface of spherical type, there exists three closed geodesics.*

Imagine in your mind that you are at the see shore and you pick up a very smooth stone. Take a very thin and very elastic rubber band and stretch it over the stone in such a way that it "does not slip." If you succeed, you found yourself a geodesic. On the spherical ball, the geodesics are the great circles. If the rubber band moves a bit from a great circle it will slip off the ball. On the ellipsoid there are only three geodesics: These are formed by intersections of the ellipsoid with the planes perpendicular to it's axes. Poincare conjectured that "on an arbitrary smooth stone, there are at least three geodesics." In 1929 this conjecture was proved by Liusternik and Schnirel'mann. That created a world wide sensation.

Theorem 3. *There exists a natural number N, such that any natural number is a sum of at most N prime numbers*.

Problem. *Any natural number, larger or equal than six, can be represented as a sum of three prime numbers*.

This question was presented to Euler by Christian Goldbach. He was a German mathematician who lived half of his life in Russia and who actually died in Moscow. He posed this question in a letter from June 7, 1742. In reply, dated June 30, 1942, Euler showed that in order to solve this problem it is enough to show that an arbitrary even number, larger or equal to 4, is a sum of two primes.

The first progress in search of the solution of this problem (not solved until the present day) was made by Schnirel'mann. (Prior to that time, Hardy and Littlewood published a paper in which they prove the Goldbach hypothesis (for sufficiently large natural numbers), under an assumption of the truth of some (heretofore unproven) hypothesis. In 1937 I. M. Vinogradov proved the Goldbach conjecture for sufficiently large natural numbers.) However, the most important thing here was not the fact that any natural number can be represented by a sum of bounded number of prime terms. (In fact, in the Schnirel'mann's method, the number of summands is estimated to be several hundred thousands.) What was the most important was actually the highly original and ingenious method, which can be extended to solve several other problems. We will say a few words about this later on.

In 1931 Schnirel'mann was send abroad for three months, and the trip was a tremendous success. For a period of time he was working in Göttingen – the Mecca of mathematics in those days – where the great Hilbert lived and worked. (Schnirel'mann is remembered not only by his phenomenal results, but also by the fact that "walked barefoot through the streets of Göttingen" – as Constance Reid wrote in her book about Courant.) He was asked by the most prestigious German publisher to write a monograph, but this was not to be: German government was taken over by fascists.

In 1933 Schnirel'mann was chosen to be a corresponding member of Academy of Sciences of the USSR.

In 1934 the governing board of the Moscow Mathematical Society decided to conduct the first high school mathematical Olympiad and L. G. Schnirel'mann was appointed to the organizing committee. He was one of the initiators (together with Liusternik and Gel'fand) of the mathematical circles at the MGU. The professors and the lecturers from that institution would twice a month give talks to the high school students. Schnirel'mann was also one of the organizers of these talks. In particular, he would give lectures on the higher dimensional geometry and group theory.

Schnirel'mann was one of the first people in Moscow to study convex geometry. He wrote a remarkable paper on the subject of applications of convex geometry to the theory of the best approximations. It was published posthumously.

One should also mention one other paper of his, written jointly with L. S. Pontriagin, on the “metric definition of dimension.” This paper had an influence on the development of the concept of ϵ-entropy, carried out by Kolmogorov.

Lev Genrikhovich was very friendly with Liusternik, Gel'fond, and Gel'fand. Many people recall him as an individual of extremely high caliber, a person who was sensitive and delicate, who had varied intellectual interests, a person of sharp mind, a keen observer, and who was also very spiritual.

His life ended tragically: On the 24th of September 1938 he committed suicide. The people of the older generation, with whom we had an opportunity to discuss the matter, connect the tragedy to the bloody and senseless atmosphere of those times: They say that NKVD became interested in Lev Genrikhovich. This made him extremely frightened, and he decided to end his life. Perhaps the veil of mystery over those events will be lifted one day, when someone seeking the truth will have an opportunity to look at the archives of KGB.

We now come to a discussion of various theorems proved by Schnirel'mann. We begin with a result from his young age, which we already mentioned.

Theorem about partition of a sphere. *Suppose a sphere is painted with three distinct colors. Then, there exists a pair of antipodal points (i. e., diametrically opposed points) which are painted the same color*.

This formulation needs to be made a bit more precise. Painting the sphere S with three colors means decomposing the sphere into three parts $F_1, F_2, F_3,$ the union of which is S. Moreover, we do not require that these sets are disjoined, each point can be simultaneously painted with several colors. However, without additional assumptions on the sets $F_i,$ the assertion is obviously false: One can decompose the sphere into two non-intersecting parts F_1 and F_2 such that for any pair of antipodal points x, y, one of the points x, y belongs to F_1 and the other to F_2. However, for an arbitrary such a decomposition, the sets F_1 and F_2 turn out to be not closed: One of the sets contains a non-empty part of the boundary of the other. (A set F in the plane or in the space is called closed if it contains its boundary. Equivalently, every point not belonging to the set is located a positive distance from the set, and is not immediately adjacent to it.)

Now we can give a correct formulation of the theorem about decomposition of the sphere.

Theorem 4. *Suppose a sphere is covered by three closed sets. Then one of these sets contains a pair of antipodal points*.

What is meant here by a sphere is the ordinary two dimensional sphere S^2, located in the three dimensional space $\mathbb{R}^3$, and which is given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$. Analogously, one defines a sphere S^n for an arbitrary natural number n: It is lies in the

$(n+1)-$ dimensional space $\mathbb{R}^{(n+1)}$ and consists of the set of solutions of the equation

$$\sum_{i=1}^{n+1} x_i^2 = 1$$

If $(x_1, x_2, \ldots, x_n)$ is a point of the n-dimensional sphere, then its antipode is the point $(-x_1, -x_2, \ldots, -x_n)$. Schnirel'mann proved his theorem for spheres of arbitrary dimension: If an n-dimensional sphere is painted in $n+1$ colors (i. e., it is covered by closed sets $F_1, \ldots, F_{n+1}$), then one can find a pair of antipodal points which are painted the same color.

Schnirel'mann's theorem is equivalent to another theorem, proved in 1930's by two Polish mathematicians K. Borsuk and S. Ulam: Every continuous transformation f of the sphere S^n into the space $\mathbb{R}^n$, glues together two antipodal points. Another words, there is a point $x \in S^n$ such that $f(x) = f(-x)$. (Here, and in what follows, all the transformations are assumed to be continuous.) One more, equivalent form of Borsuk-Ulam theorem is this: *There exists no odd transformation* $f : S^n \mapsto S^{n-1}$. A transformation f is called odd if $f(-x) = -f(x)$ for all x's.

Let us now explain why these three theorems are equivalent. We will denote them by Ш (Schnirel'mann's theorem about painting of spheres), БУ_1 (the theorem about gluing of antipodal points), and БУ_2 (the theorem about the nonexistence of an odd transformation). Since $S^{n-1} \subseteq \mathbb{R}^n$, any transformation into S^{n-1} can be viewed as a transformation into $\mathbb{R}^n$. In this way, the implication $\text{БУ}_1 \Rightarrow \text{БУ}_2$ is obvious. Conversely, suppose БУ_2 holds. Let us assume that $f : S^n \mapsto \mathbb{R}^n$ is a counterexample for БУ_1, i. e., f a transformation which does not glue antipodal points. Then, the transformation $g : S^n \to \mathbb{R}^n$given by the formula $g(x) = f(x) - f(-x)$ is odd and does not assume the value zero. There exists a natural transformation $r : \mathbb{R}^n/\{0\} \mapsto S^{n-1}$, which assigns to every non-zero vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ a unit vector $r(x)$ which has the same direction as x:

$$r(x) = \frac{x}{\|x\|}$$

where $\|x\| = \left(\sum x_i^2\right)^{\frac{1}{2}}$. The composition $rg : S^n \mapsto S^{n-1}$ is odd, contrary to БУ_2.

We now establish the equivalence of the Borsuk-Ulam and Schnirel'mann's theorems. We start with БУ_1 and let $F_1, \ldots, F_{n+1}$ be closed subsets of the sphere S^n, such that their union is S^n. We must show that for some i, $1 \leq i \leq n+1$, the set F_i contains a pair of antipodal points. If there exists a point x, belonging to all the sets F_i, the situation is clear: Some set F_i contains the antipodal pair $x, -x$. Assume now that $F_1 \cap \ldots \cap F_{n+1}$ is empty. Then, for each $x \in S^n$, let $f_i(x)$ be the distance of the point x to the set F_i. Hence $f_i : S^n \mapsto \mathbb{R}$ is a positive, non-negative function, and $f_i(x) = 0$ if and only if $x \in F_i$. (Here we use the hypothesis that the sets F_i) are closed. As a consequence of our hypothesis, the functions f_i, $1 \leq i \leq n+1$, never assume the value

0 simultaneously, thus the function

$$h = \sum_{i=1}^{n+1} f_i$$

is everywhere positive. Define $g_i = \frac{f_i}{h}$, $1 \leq i \leq n+1$, and

$$G(x) = (g_1(x),\, \ldots, g_n(x))$$

Then $G : S \mapsto \mathbb{R}^n$ is a continuous transformation. We apply the Borsuk-Ulam theorem to G and conclude that there is $x \in S^{n+1}$, such that $g_i(x) = g_i(-x)$ for each $i = 1, \ldots, n$. Since

$$g_{n+1} = 1 - \sum_{i=1}^{n+1} g_i$$

we also have $g_{n+1}(x) = g_{n+1}(-x)$. If i is such that $x \in F_i$, then $g_i(x) = g_i(-x) = 0$, thus F_i contains a pair of antipodal points.

We preface the proof of the implication $\text{Ш} \Rightarrow \text{БУ}_2$ with the following remark: In the Schnirel'mann's theorem, the number $n+1$ of the colors cannot be replaced by $n+2$. Another words, the n-dimensional sphere can be covered with a collection of closed sets $F_1,\, \ldots, F_{n+2}$, none of which contains a pair of antipodal points. Let, for example

$$F_i = \{(x_1,\, \ldots, x_{n+1}) \in S^n : \, x_i \geq \epsilon\},\, i = 1,\, \ldots, n+1$$

and

$$F_{n+2} = \left\{(x_1, \ldots, x_{n+1}) \in S : \sum_{i=1}^{n+1} x_i \leq -\epsilon\right\}$$

For sufficiently small ϵ, the sets $F_1, \ldots F_{n+2}$ cover the sphere, and none of them contains a pair of antipodal points. We are now ready to prove implication $\text{Ш} \Rightarrow \text{БУ}_2$. Suppose that $f : S^n \mapsto S^{n-1}$ is an odd transformation. If follows from the previous remarks that the sphere S^{n-1} can be covered by closed sets $F_1 \ldots, F_{n+1}$, none of which contains a pair of antipodal points. Denote by $f^{-1}(A)$ to be pre-image of the set A, i. e., the set of points of those x, for which $f(x) \in A$. Then, $f^{-1}(F_1), \ldots, f^{-1}(F_{n+1})$ is a cover of the sphere S^n by closed sets, none of which contains a pair of antipodal points. This is impossible by the Schnirel'mann's theorem.

Let us sketch the proof of the Borsuk-Ulam theorem (in the form БУ_2) in the case $n = 2$. We will show that there does not exist an odd transformation $f : S^2 \mapsto S^1$. By the arguments above, this will also prove the Schnirel'mann's theorem about painting a two dimensional sphere with three colors.

We remark that for $n = 1$ the Borsuk-Ulam theorem (in the form БУ_2) is trivial, since the zero-dimensional sphere consists of two pints ± 1, and a continuous transformation of S^1 into S^0 must be a constant, we cannot “tear” the circle into two parts. We will reduce the proof of $БУ_2$ for $n = 2$ to the case $n = 1$.

Suppose a point is moving on a circle, and after a while returns to the original position. Intuitively, it is clear what is meant by a *number of complete winds (circumnavigations)*, or the *winding number*, the point makes during the entire course of the motion (even if the movement of the point is not always in the same direction). Formally, we can define this number as follows. Suppose the motion of the point on the circle is described by a continuous function $f : I \mapsto S^1$, where $I = [a, b]$ is an interval on the real line. Then there exists a continuous function $\varphi : I \mapsto \mathbb{R}$ such that

$$f(t) = (\cos\varphi(t), \sin\varphi(t))$$

for all $t \in I$. Such a function is determined uniquely, up an addition of a constant of the form $2\pi k$. This means that the difference $\varphi(b) - \varphi(a)$ is uniquely determined. If $f(b) = f(a)$, then $(\varphi(b) - \varphi(a))/2\pi$ is an integer, which is then called the winding number.

Suppose now we are given a continuous transformation $f : S^1 \mapsto S^1$ of a circle into itself. We change this transformation into a transformation $g : [0, 2\pi] \mapsto S^1$, by putting $g(t) = f(\cos t, \sin t)$. The number of winds the point $g(t)$ makes when t varies from 0 to 2π is called the *degree* of the transformation f. For example, the identity transformation has degree 1, the constant transformation has degree 0, and the symmetry transformation about any diameter of the circle has degree -1. Since the degree is an integer, it cannot change under a continuous deformation of the transformation. (We will not prove this assertion, nor will we give an exact definition what is a deformation of a transformation.)

Suppose D is the closed disk bounded by the circle S^1. We have the following

Proposition 1. If the transformation $f : S^1 \mapsto S^1$ can be extended to a continuous transformation $F : D \mapsto S^1$, then the degree of f is zero.

Proof. For $r \in [0, 1]$ and $x \in S^1$ set $f_r(x) = F(rx)$. The transformation $f_r(x)$ depends on r continuously, so that all the transformations f_r have the same degree. Since f_0 is a constant transformation, it has degree zero. Consequently, the degree of $f_1 = f$ is also 0.

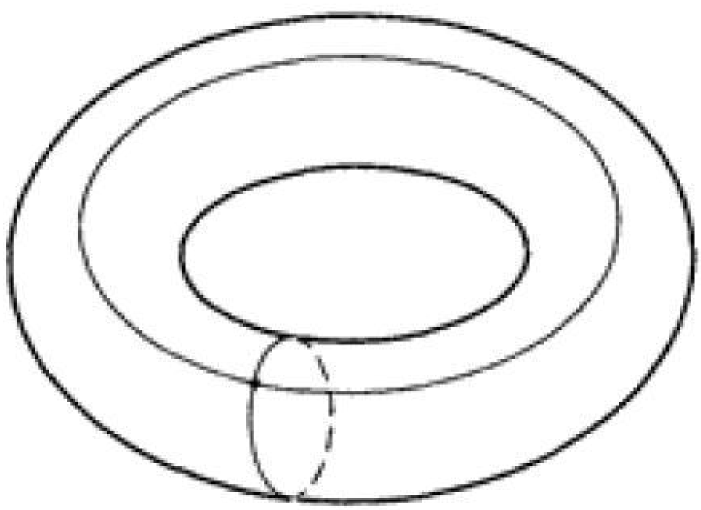
Proposition 2. If $f : S^1 \mapsto S^1$ has degree zero, then f glues together some pair of antipodal points (i. e., there is a point $x \in S^1$ such that $f(x) = f(-x)$).

Proof. From the hypothesis one can easily deduce that there exists a continuous function $\varphi : I \mapsto \mathbb{R}$ such that $f(x) = (\cos\varphi(x), \sin\varphi(x))$ for all $x \in S^1$. Since φ glues together a pair of antipodal points (case $n = 1$ of the Borsuk-Ulam theorem), the same is true for f.

The case $n = 2$ of the Ulam-Borsuk theorem follows now directly from the Propositions 1 and 2. It is enough to show that any transformation $F : S^2 \mapsto S^1$ glues a pair of antipodes. Consider the restriction f of the transformation F to the circle S^1, resulting from the intersection of the sphere with the horizontal plane. We identify this circle with S^1, and identify the resulting disk with the disk D from the Proposition 1. Suppose p is the projection of the upper hemisphere onto D, and q be the inverse of p. Then, the transformation $x \mapsto F(q(x))$ from D into S^1 is an extension of f. It follows from the Proposition 1 that f has degree zero. It follows then from the Proposition 2 that f glues a pair of antipodal points. Thus we have proved the Borsuk-Ulam-Schnirel'mann theorem for a two-dimensional sphere.

We will now present another remarkable example, due to Schnirel'mann, of an application of topological methods. What we have in mind is the theorem about inscribed square. We already have encountered the statement of this theorem: *In any closed curve one can inscribe a square*. We now present the idea of the proof.

Consider the space of all quadrilaterals in the plane. Since each such quadrilateral can be described by eight numbers, we can say that this set is *eight-dimensional*. We now consider two “four-dimensional” subsets A and B as follows: Each element of A and B will be described by four parameters. The set A will consist of all squares, and the set B will consist of all the quadrilaterals whose vertices lie on the curve. The theorem asserts that these two sets intersect. We will begin with an assumption that the curve is an ellipse. In this case, it is easy to show that there exists exactly one inscribed square. Thus, the sets A and B, when defined for an ellipse, intersect. In the general case, one can continuously deform the give curve into an ellipse.



The set B will also deform continuously during that operation. One can show that in the case of ellipse, the sets A and B intersect in a “robust”: that intersection cannot disappear under a continuous deformation. One can make an analogy with the intersection of the meridian and the parallel on the torus (the surface of a “doughnut” – see the illustration). Consequently, $A$and B intersect for an arbitrary curve.

Carrying out the details of this idea presents several difficulties. I order to apply the results of the corresponding theory it is necessary to assume that the sets A and B include "degenerate" squares, i. e., those for which all the vertices collapse to a single point. How does one avoid the possibility that the intersection $A \cap B$ will contain only such degenerate points? For this purpose Schnirel'mann assumes that the curve is sufficiently smooth (is actually twice differentiable). Quite often, when this Schnirelman theorem is cited, it is formulated for the general curve, without this extra hypothesis. The authors don't know if the proof in general case was ever published.

Finally, we will describe **Schnirel'mann's method in the additive number theory.** Suppose A and B are two sets of natural numbers. The *sum* of A and B is the set, denoted by $A + B$, of all the numbers of the form $a + b$, where $a \in A$, $b \in B$. For our purpose it will be more convenient to define the sum of the sets A and B as the set given by $A \oplus B =$ (A+B) $\cup A \cup B$, i. e., the set obtained from $A + B$ by adding all the elements of A and B. We say that a set A is a basis of the natural numbers if for some k the $k-$ fold sum$A \oplus \ldots \oplus A$, coincides with the set of all natural numbers. For example, if A is the set of all squares, then A is a basis. This follows from the well known theorem, due to Lagrange, which says that any natural number is a sum of at most four squares. This means that $A \oplus A \oplus A \oplus A = \mathbb{N}$. Let P be the set of all prime numbers. Is P a basis? Schnirel'mann was the first one to give a positive answer to this question. He proved that indeed P *is a basis*. We now present the main ideas of that proof.

We follow Schnirel'mann and introduce a concept of the *density* of a set A of natural numbers. For each $n \in \mathbb{N},$ let $A(n)$ be the number of elements of the set A which are in the interval $[1, n]$. We define the *density* $d(A)$ of the set A to be the lower bound of the numbers $A(n)/n$ taken over all $n \in \mathbb{N}$ (i. e., the largest number a such that $A(n)/n > a$ for an arbitrary n). Another words, the density is the largest number α, such that $A(n) \geq \alpha n$ for all $n \in \mathbb{N}$. Schnirel'mann proves the following result:

Theorem 5. *Any set of natural numbers having a positive density is a basis*.

This theorem cannot be directly applied to the set P of prime numbers, augmented by the number 1, because the resulting set has density equal to zero. (Chebyshev proved that the number $\pi(n)$, of prime numbers not exceeding n, is less than $Cn/\log n,$ for some C; See the paper by V. Tikhomorov, "On the Chebyshev's theorem about the distribution of prime numbers", *Kvant*, No. 6, 1994)

However, Schnirel'mann has proved that $P \oplus P$ has a positive density, from which it follows that P is a basis. We recall, that Euler's question whether or not the set $P \oplus P$ contains all even numbers remains open.

We will prove Theorem 5. The assertion will follow from the following two lemmas.

Lemma 1. If $A, B \in \mathbb{N}$ and $d(A) + d(B) > 1$, then $A \oplus B = \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$. If $n \in B$ then $n \in A \oplus B$. If $n \notin B$ then consider two subsets of the interval $[1, n]$: $\{a \in A : a \leq n\}$ and $\{n - b : b \in B, b \leq n\}$ They are guaranteed to intersect, since the first set has at least $n \cdot d(B)$ elements, the second has at least $n \cdot d(B)$ elements and $n \cdot d(A) + n \cdot d(B) > n$. Consequently, $a = n - b$ for some $a \in A$, $b \in B$, so that $n = a + b \in A \oplus B$.

Lemma 2. (Schnirel'mann inequality) For arbitrary A and $B \subset \mathbb{N}$ we have the following

$$d(A \oplus B) \geq d(A) + d(B) - d(A) \cdot d(B).$$

Proof. Set $C = A \oplus B, \alpha = d(B)$, and $\beta = d(B)$. Fix $n \in \mathbb{N}$. We have to estimate the number $C(n)$ from below. Let $a_1 < \ldots < a_r$, $r = A(n)$ be all the elements of the set A which are also in the interval $[1, n]$. The interval $[1, n]$ is partitioned by the numbers $a_1, \ldots, a_r$ into $r + 1$subintervals (some of these may be empty) of lengths $l_1 = a_1 - 1$, $l_2 = a_2 - a_1, \cdots, l_{r+1} = n - a_r$. The k^{th} such subinterval contains at least $\beta \cdot l_k$ numbers from C. For $k > 1$ these are numbers of the form $a_{k-1} + b$, where $b \in B$, $b \leq l_k$, and for $k = 1$ these are numbers from B, that are larger than l_1. From this we deduce the following estimate:

$$C(n) \geq r + \beta \cdot \sum l_k = r + \beta(n - r) = (1 - \beta)r + \beta n \geq (1 - \beta)\alpha n + \beta n$$

which implies that $d(C) \geq (1 - \beta)\alpha + \beta = \alpha + \beta - \alpha\beta$.

We now prove the Theorem 5 from Lemmas 1 and 2. The inequality in Lemma 2 can be written as

$$1 - d(A \oplus B) \leq (1 - d(A)) \times (1 - d(B))$$

In this form it can be extended (by induction) to the arbitrary number of terms:

$$1 - d(A_1 \oplus \ldots \oplus A_n) \leq \prod_{i=1}^{n} (1 - d(A_i)).$$

Suppose now A is a set with positive density and

$$A_k = A \oplus \ldots \oplus A$$

be the sum of k terms, each one equal to A. The last inequality shows that $d(A_k)$ approaches 1 with increasing k. Let k be such that $d(A) > 1/2$. It follows from Lemma 1 that $A_{2k} = \mathbb{N}$. This proves Theorem 5.

For every $n \in \mathbb{N}$ let $W = \{1^n, 2^n, \ldots\}$ be the set of all the n^{th}powers. Is W_n a basis? This is what's called the Waring's problem. The question was answered in a positive way by Hilbert at the beginning of (the 20^{th}, tr.) century. The solution turned out to be quite complicated. Theorem 5 allows one to obtain another solution: It is sufficient to show that the k-fold sum $W_n \oplus \ldots \oplus W_n$ is of positive density for sufficiently large k. An elementary (but very complicated) solution of the Waring's problem, based on the Schnirel'mann method, can be found in the book by Khinchine [1].

In conclusion, we make several remarks.

1. From Proposition 1 we can deduce the following result:

Brouwer's Theorem. There does not exist a continuous map $F$of the disk D into the circle S^1 which fixes the points on the boundary, such that $F(x) = x$ for all $x \in S^1$.

Indeed, the identity transformation on the circle bounding the disk has degree 1, hence it cannot be extended to the mapping $F : D \mapsto S^1$. From this one can easily deduce the theorem about a fixed point: *for any transformation* $F : D \mapsto D$ *there is* $x \in D$ *such that* $F(x) = x$.

The concept of degree of the transformation can be extended to a transformation of the n-dimensional sphere, and one can use this concept to prove the fixed point theorem for the $(n+1)$-dimensional ball. This was done by Brouwer in the early part of this century (20th, tr.) and the result was a spectacular achievement of the emerging field of mathematics $-$ topology.

2. A well known geometer Boris Nikolaevich Delone, when commenting on the Schnirelman's theorem about the squares inscribed in a curve, noticed that if the curve in question is convex, the theorem can be proved by elementary methods. We encourage you to try to find such an elementary proof for yourself.

3. Suppose $A = \{1, 4, 9, 16, \dots\}$ be the sequence of all the squares of the natural numbers. We remarked that $A \oplus A \oplus A \oplus A = \mathbb{N}$. Do you know what are the sets $A \oplus A$ and $A \oplus A \oplus A$? The first set actually consists of all the integers, whose factorization into primes has the property that all the primes of the form $4k + 3$ occurring in the factorization, appear with an even exponent. The second set is obtained from the first by adding all the numbers of the form $4^a(8b + 7)$.

4. In connection with Lemma 2 let us quote (with some abbreviations) from the book [1]: “In the fall of 1931 when Schnirel'mann talked about his discussion in Göttingen with E. Landau, he reported that he established the following interesting fact: For all the examples he could think of, the inequality

$$d(A \oplus B) \geq d(A) + d(B) - d(A)d(B)$$

can be replaced with a stronger, and simpler inequality:

$$d(A \oplus B) \geq d(A) + d(B)$$

(provided that $d(A) + d(B) \leq 1$). The first attempts to prove this conjecture were unsuccessful. The problem became fashionable. The mathematical community was fascinated by it. A good half of the English mathematicians put aside whatever they were doing and were working on trying to find the proof. But, the problem turned out to be quite stubborn, and for many it years resisted all the attempts, some of which used the most sophisticated of methods. Only in 1942 it was finally cracked by a young American mathematician Mann.”

The proof of the Landau-Schnirel'mann conjecture can be found in the Khinchin's book [1]. We strongly encourage the readers to get acquainted with this remarkable book.

Equally deserving your attention is the book by Schnirel'mann himself [2]. You can find there the proof of Lagrange's theorem about the sum of four squares, the proofs of the great Fermat theorem for the exponents 3 and 4, as well as many other things.

References

1. Khinchin, A. Ja., Three pearls of number theory
2. Schnirel'mann, L. G., Prime numbers.

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