

Markov numbers

The best introduction and discussion of the subject is in [4] Cassels, An Introduction to Diophantine Approximation, Chapter 2.

Markov numbers are defined to be the integers occurring as solutions of the Diophantine equation:

$$x^2 + y^2 + z^2 - 3xyz = 0 \quad (1)$$

When talking about a triplet satisfying the equation (1), say (x_0, y_0, z_0) , we will always assume that

$$x_0 \leq y_0 \leq z_0 \quad (2)$$

Theorem 1 : The only triplets satisfying (1), in which an equality occurs in (2), are $(1, 1, 1)$ and $(1, 1, 2)$

Proof.

(α) Assume $x_0 = y_0$. This leads to the equation in z :

$$2x^2 + z^2 - 3x^2z = 0$$

Solving for z , we get

$$z = \frac{1}{2} \left(3x^2 \pm \sqrt{9x^4 - 8x^2} \right) = \frac{1}{2} \left(3x^2 \pm x\sqrt{9x^2 - 8} \right)$$

This requires that $9x^2 - 8$ be a perfect square, say t^2 . Hence

$$9x^2 - 8 = t^2 \text{ or } 9x^2 - t^2 = 8$$

or

$$(3x - t)(3x + t) = 8 = 1 \cdot 8 = 2 \cdot 4$$

This leads to two possibilities:

$$(3x - t = 1 \text{ and } 3x + t = 8) \Rightarrow 6x = 9 \text{ i.e. no integer solution}$$

and

$$(3x - t = 2 \text{ and } 3x + t = 4) \Rightarrow x = 1$$

To solve for z , we plug $x = y = 1$. Then (1) becomes $2 + z^2 - 3z = 0$, giving $z = 1$ or $z = 2$.

(β) Assume $y_0 = z_0$. This leads to the equation z

$$2y^2 + z^2 - 3y^2z = 0$$

which is identical with case (α), except that y replaces x . So the only solutions in this case are also (1, 1, 1) and (1, 1, 2) QED.

Assume (x, y, z) is a solution of (1), with $x \leq y \leq z$. If we keep two of the variables fixed, this produces a quadratic equation in the third variable, with two solutions. Since the equation is monic with integer coefficients, the other solution is also an integer, call it x' , y' , or z' , whichever is the case.

If

$$t^2 + bt + c = 0$$

is a quadratic equation in t , with roots r_1 , and r_2 then

$$\begin{aligned} r_{1,2} &= \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4c} \right) \\ r_1 + r_2 &= -b \\ r_1 \cdot r_2 &= c \end{aligned}$$

Thus, we have the following formulas:

$$x' = 3yz - x = \frac{y^2 + z^2}{x} = \frac{1}{2} \left(3yz \pm \sqrt{9y^2z^2 - 4(y^2 + z^2)} \right) \quad (3)$$

$$y' = 3xz - y = \frac{x^2 + z^2}{y} = \frac{1}{2} \left(3xz \pm \sqrt{9x^2z^2 - 4(x^2 + z^2)} \right) \quad (4)$$

$$z' = 3xy - z = \frac{x^2 + y^2}{z} = \frac{1}{2} \left(3xy \pm \sqrt{9x^2y^2 - 4(x^2 + y^2)} \right) \quad (5)$$

For example, the triplet of solutions (1, 2, 5) leads to three new solutions:

$$\begin{aligned} (2, 5, 29) &\text{ where } x' = 29 \\ (1, 5, 13) &\text{ where } y' = 13 \\ (1, 1, 2) &\text{ where } z' = 1 \end{aligned}$$

We can actually figure out the signs in the formulas (3), (4), and (5):

Theorem 2. Assume that $x < y < z$. In the formulas (3) and (4) the sign is $+$, and in the formula (5) the sign is $-$.

Proof: This will follow from the following lemma:

Lemma 1. Suppose $1 \leq t < z$ and $1 \leq x < y$ are integers. Then

$$(\alpha) \quad \frac{1}{2} \left(3tz + \sqrt{9t^2z^2 - 4(t^2 + z^2)} \right) \geq z$$

and

$$(\beta) \quad \frac{1}{2} \left(3xy - \sqrt{9x^2y^2 - 4(x^2 + y^2)} \right) < y$$

Proof: Part (α) is clear:

$$\frac{1}{2} \left(3tz + \sqrt{9t^2z^2 - 4(t^2 + z^2)} \right) > \frac{3tz}{2} > z$$

As to part (β) , let $\psi(x, y) = \frac{1}{2} \left(3xy - \sqrt{9x^2y^2 - 4(x^2 + y^2)} \right)$.

$$\psi(x, y) = \frac{1}{2} 3xy \left(1 - \sqrt{1 - \frac{4}{9} \left(\frac{1}{x^2} + \frac{1}{y^2} \right)} \right)$$

or, after a bit of algebra:

$$\psi(x, y) = \frac{1}{1 + \sqrt{1 - \frac{4}{9} \left(\frac{1}{x^2} + \frac{1}{y^2} \right)}} \cdot \frac{1}{2} \cdot 3xy \cdot \frac{4}{9} \left(\frac{1}{x^2} + \frac{1}{y^2} \right)$$

Now, the first fraction, the one with the square root in the denominator, assumes maximum when $x = 1$ and $y = 2$, because of the hypothesis on x and y . Thus, the fraction in question at most $\frac{3}{5}$. Hence

$$\psi(x, y) \leq \frac{3}{5} \cdot \frac{1}{2} \cdot 3 \cdot \frac{4}{9} \cdot \left(\frac{y}{x} + \frac{x}{y} \right) \leq \frac{2}{5} (y + 1) < y. \text{ QED.}$$

Proof of Theorem 2: If "-" holds in (3) or (4), then

$$x \text{ or } y = \frac{1}{2} \left(3tz + \sqrt{9t^2z^2 - 4(t^2 + z^2)} \right)$$

where $t = y$ or x , which would imply that $x > z$ or $y > z$, contrary to hypothesis.

If "+" holds in (5) then

$$z = \frac{1}{2} \left(3xy - \sqrt{9x^2y^2 - 4(x^2 + y^2)} \right) < y$$

again contradicting the hypothesis.

Definition 1. Let (x, y, z) be a solution of (1) satisfying (2). Define

$$\begin{aligned} X(x, y, z) &= (y, z, x') \\ Y(x, y, z) &= (x, z, y') \\ Z(x, y, z) &= (x, z', y) \text{ or } (z', x, y) \end{aligned}$$

where x' , y' , and z' are given by the formulas (3), (4), and (5).

Theorem 3. If $x < y < z$ then

$$z < y' < x' \text{ and } z' < y$$

Proof: The facts that $z < y'$ and $z' < y$ are proved in Lemma 2. To show that $y' < x'$ we have by (3) and (4):

$$x' - y' = (3yz - x) - (3xz - y) = (3z + 1)(y - x) > 0. \text{ QED.}$$

Definition 2. If (x, y, z) is a solution of (1) satisfying $x \leq y \leq z$, define

$$\|(x, y, z)\| = z$$

i.e., the largest of the numbers x , y , and z .

Theorem 4. (Markov [9]) Every solution of (1) can be obtained by starting with the triple $(1, 1, 1)$ and repeatedly obtaining new solutions by one of the transformations X and/or Y .

Proof. One easily checks that all the solutions (x, y, z) with $z < 10$, say, can be so obtained. Given a solution (x, y, z) , with $z \geq 10$, we have

$$\begin{aligned} Z(x, y, z) &= (a, b, c) \text{ where} \\ (a, b, c) &= (x, z', y) \text{ or } (z', x, y) \end{aligned}$$

Now, $c = y < z$, and $(x, y, z) = X(a, b, c)$ or $(x, y, z) = Y(a, b, c)$. Continuing in this manner we get to a triple (a, b, c) with $c < 10$, and the result follows. QED.

There is a conjecture due to Frobenius which is almost a century old, see [6]:

Given a Markov number z , the sequence of transformations of X 's and Y 's leading from $(1, 1, 1)$ to (x, y, z) is unique.

Another way of restating this conjecture is:

If (x_1, y_1, z) and (x_2, y_2, z) are two solutions of (1), with $x_i \leq y_i \leq z$, then $x_1 = x_2$ and $y_1 = y_2$.

It is known that this is true when z is a power of a prime. See [1], [2], and [3] for the discussion and the proofs.

Markov numbers arise in the theory of approximations as follows. (See Cassels [4] for a complete discussion.) Suppose θ is a positive irrational number whose continued fraction expansion is:

$$\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots \frac{1}{a_n + \dots}}}} \quad (6)$$

which we will denote as

$$\theta = [a_0; a_1, a_2, \dots, a_n, \dots] \quad (7)$$

Assume that the "continued fraction digits" a_i of θ are bounded. Then there is a number $M(\theta)$ with the following properties:

(α) For every $\epsilon > 0$ there are infinitely many pairs of integers p and q such that:

$$\left| \theta - \frac{p}{q} \right| < \frac{1 + \epsilon}{M(\theta)q^2}$$

(β) For every $\epsilon > 0$ there are only finitely many pairs of integers p and q such that

$$\left| \theta - \frac{p}{q} \right| < \frac{1 - \epsilon}{M(\theta)q^2}$$

In fact

$$M(\theta) = \limsup_{n \rightarrow \infty} [a_{n+1}; a_{n+2}, \dots] = \limsup_{n \rightarrow \infty} a_n + \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \frac{1}{\dots}}}$$

Moreover, let

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_n}}} \quad (8)$$

be the continued fraction of θ truncated at a_n . The integers p and q in (α) are then actually given by (8).

The totality of all the numbers $M(\theta)$, where θ 's is taken to be all irrationals with bounded continued fraction digits, is called **Markov spectrum**. Denote it by M . Its structure is topologically quite complicated, see Cusick and Flahive [5] for a comprehensive discussion. The set M has many isolated points, in fact if

$$u_1, u_2, u_3, u_4, \dots = 1, 2, 5, 13, 29, \dots$$

is the sequence of all Markov numbers arranged in increasing order, then

$$\mu_n = \sqrt{9 - \frac{4}{u_n}}$$

is an isolated point of Markov spectrum. The numbers $\mu_n \uparrow 3$ and all the other points μ of the spectrum M satisfy $\mu \geq 3$. The structure of the set M above 3 is very complicated, in places it resembles the Cantor's set. There is however a point x , such that M contains all the numbers $\geq x$. See [5] and [7].

References:

- [1] A. Baragar, On the unicity conjecture for Markoff numbers, *Canad. Math. Bull.* 39 (1996), 3 – 9.
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